J. Fluid Mech. (1978), *vol.* 88, *part* 2, *pp.* 309–321
Printed in Great Britain

On solutions of the boundary-layer equations with algebraic decay

By J. H. MERKIN
Department of Applied Mathematical Studies, University of Leeds, England

(Received 29 July 1977 and in revised form 16 March 1978)

The boundary-layer equations are solved numerically for mainstreams

$$U(x) = x(1-x^2)^{-\alpha} \quad \text{and} \quad U(x) = (1-x)^{-\alpha},$$

which are both $O((1-x)^{-\alpha})$ near $x = 1$. Series expansions are derived near $x = 1$. For $\alpha > 1$, where, for the similarity solution at $x = 1$, the outer boundary condition is approached through exponentially small terms, a straightforward expansion in powers of $1-x$ is possible. For $0 < \alpha < 1$, where the decay is only algebraic (Brown & Stewartson 1965), the outer boundary condition cannot be satisfied even with algebraic decay by the higher-order terms in the series and this must be regarded as only an inner expansion. An outer expansion is required which matches with this inner expansion and which approaches the outer boundary condition with exponential decay. For $\alpha = 1$, the decay is exponential, but not of the same form as for $\alpha > 1$, and again the outer boundary condition cannot be attained by the higher-order terms in the series. An outer expansion for this case is also derived.

1. Introduction

In his work on the viscous incompressible flow in a cone, Ackerberg (1965) found, when considering the flow for small viscosity, that near the apex the solution for the boundary layer on the wall of the cone approached the outer core flow through algebraically small terms and not through the usual exponentially small terms. He then argued that, since this asymptotic behaviour was unacceptable, this core flow would not be the potential sink as originally suggested, but that near the apex there would be a vortex motion with closed streamlines which the inner boundary-layer solution could then approach with an exponentially small error. This problem was considered in more detail by Goldstein (1965), who obtained similarity solutions of the boundary-layer equations for (non-dimensional) mainstreams $U(x)$ of the form $U(x) = (1-x)^{-\alpha}$, where x measures distance along the wall. Ackerberg's case corresponds to $\alpha = \frac{2}{3}$. He showed that there could be solutions with exponential decay for $\alpha \geqslant 1$, solutions with algebraic decay for $0 < \alpha < 1$ and no solutions for $\alpha \leqslant 0$. He concluded that the solutions with algebraic decay were unacceptable in the context of the usual boundary-layer theory as they could not be matched with the outer expansion over a finite part of the x axis.

Brown & Stewartson (1965) pointed out that solutions of the boundary-layer equations with algebraic decay could be allowed and would not contradict Goldstein's argument provided that they held only at singular points of the equations and not over a finite range of x. They showed that this type of similarity solution could be a limit of a solution of the full boundary-layer equations as $x \to 1$ but that, in this case, this limit would not commute with the limit $y \to \infty$, where y measures distance normal to the wall in the usual boundary-layer variables. A non-commutative limit of this type has also been reported by Buckmaster (1969) in his work on the flow at the rear of a cylinder when separation has been completely suppressed by magnetohydrodynamical effects.

This paper extends the work of Brown & Stewartson (1965) in two ways. First, a numerical solution of the boundary-layer equations is obtained for mainstreams which are $O((1-x)^{-\alpha})$ near $x = 1$. This confirms their result that, for all the values of α considered, the full solution does have the form given by the similarity solution at $x = 1$. They used a Görtler (1957) expansion from $x = 0$ and had to limit their attention to a mainstream for which the expansion is thought to be convergent. However, the numerical solution does not require this and the mainstreams considered here are $U(x) = x(1-x^2)^{-\alpha}$ (the one treated by Brown & Stewartson) and $U(x) = (1-x)^{-\alpha}$ (for which the Görtler series is thought not to converge as far as $x = 1$).

Second, a series expansion for the solution near $x = 1$ is derived and it is found that there are three cases to consider. For $\alpha > 1$ the limits $x \to 1$ and $y \to \infty$ commute, there is a straightforward expansion in powers of $1-x$, the leading term being the similarity solution as derived by Goldstein (1965), and all the terms in the expansion approach their mainstream values with exponential decay. For $0 < \alpha < 1$ the situation is more complicated. Here the limits $x \to 1$ and $y \to \infty$ do not commute and, as expected, when a straightforward expansion is tried with the similarity solution as the leading term, which now has only algebraic decay, this breaks down. It is found that the term $O((1-x)^n)$ in the expansion is $O(\eta^{(2n-4\alpha)/(1-\alpha)})$ for large η, where

$$\eta = (\tfrac{1}{2}A_0)^{\frac{1}{2}}\, y/(1-x)^{\frac{1}{2}(1+\alpha)}, \quad A_0 = \lim_{x\to 1}(1-x)^{\alpha}\, U(x),$$

is the similarity variable used by Goldstein, so for $n \geqslant 2\alpha$ the outer boundary condition cannot be satisfied at all. To resolve this difficulty an outer expansion is required which uses $\tau = (\tfrac{1}{2}A_0)^{\frac{1}{2}}\, y/(1-x)^{\alpha}$ as the independent variable. This choice of variable is suggested by the work of Brown & Stewartson, who showed that, for large y, the streamwise velocity component u is of the form

$$u \sim U(x) + A(x,y)\exp\left(-y^2/2F(x)\right), \tag{1}$$

where

$$F(x) = 2\int_0^x U(s)\,ds/U(x)^2$$

and

$$A \sim \frac{\beta}{U(x)\left(\int_0^x U(s)\,ds\right)^{\frac{1}{2}}}\left(\frac{U(x)\,y}{\int_0^x U(s)\,ds}\right)^m,$$

for some constants β and m. When $U(x)$ is $O((1-x)^{-\alpha})$ near $x = 1$, $F(x)$ is $O((1-x)^{2\alpha})$ for $0 < \alpha < 1$, whereas, for $\alpha > 1$, $F(x)$ is $O((1-x)^{1+\alpha})$ so the similarity variable η is recovered.

It is found that each term in the inner expansion contributes to the leading term in the outer expansion so it is not possible to determine it completely. However, an expansion can be found which matches with the inner solution and has the exponential decay given by (1) for large y. This contributing to the leading term in an outer expansion by each term in an inner expansion is not unknown in boundary-layer theory. It appears, for example, in the Goldstein–Stewartson theory of separation (Stewartson 1970).

Finally, there remains the case $\alpha = 1$. Here the similarity solution has exponential decay but of the form $\exp[-2^{\frac{1}{2}}y/(1-x)]$. Thus the limits $x \to 1$ and $y \to \infty$ do not commute and an expansion in powers of $1-x$ breaks down at the term $O((1-x)^2)$, again because it is not possible to satisfy the outer boundary condition at this stage. An outer expansion is then required in which $\zeta = y/(1-x)[-\log(1-x)]^{\frac{1}{2}}$ is used as the independent variable. This choice of variable comes from (1), for, with $\alpha = 1$, $F(x)$ is $O(-(1-x)^2\log(1-x))$ near $x = 1$. This fact was not commented on by Brown & Stewartson. In this case it is possible to determine the leading term in the outer solution completely and this has the exponential decay given by (1).

2. Numerical solution

The equations describing the two-dimensional steady flow in the boundary layer of an incompressible fluid with mainstream $U(x)$ are, in non-dimensional form,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = U\frac{dU}{dx}+\frac{\partial^2 u}{\partial y^2}, \tag{3}$$

where x and y are measured along and perpendicular to the wall respectively and u and v are the corresponding velocity components. The boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0; \quad u \to U(x) \quad \text{as} \quad y \to \infty. \tag{4}$$

Two forms of the mainstream $U(x)$ are considered, namely $U(x) = x(1-x^2)^{-\alpha}$ and $U(x) = (1-x)^{-\alpha}$.

Equations (2) and (3) were solved numerically using essentially the same method as has been described by the author elsewhere (Merkin 1972). To do this the equations had first to be transformed into a more appropriate form. For $U(x) = x(1-x^2)^{-\alpha}$ the transformation used was to write the stream function ψ (defined from (2) in the usual way) as

$$\psi = \begin{cases} x\,F(x,y) & \text{in} \quad 0 \leqslant x \leqslant \frac{1}{2}, \\ (2\xi)^{\frac{1}{2}(1-\alpha)}f(\xi,\eta) & \text{in} \quad \frac{1}{2} < x \leqslant 1, \end{cases}$$

where $\xi = 1-x$ and $\eta = y/(2\xi)^{\frac{1}{2}(1+\alpha)}$ as suggested by the similarity solution at $x = 1$. The values of the velocity components calculated at $x = \frac{1}{2}$ in the first part of the integration were used as starting values for the second part of the calculation. For $U(x) = (1-x)^{-\alpha}$ a transformation was used which enabled the integration to proceed from $x = 0$ to $x = 1$ without changing the form of the equations and which had the appropriate similarity form at $x = 0$ and $x = 1$. This was achieved by putting $\psi = (2x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}(1-\alpha)}G(x,\bar{\eta})$, where $\bar{\eta} = y/(2x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}(1+\alpha)}$.

The method involves differencing the x derivatives and averaging all the other terms over the step from x to $x+\Delta x$. This results in a nonlinear ordinary differential equation which is then written in terms of finite differences, the resulting nonlinear algebraic equations being solved iteratively by the Newton–Raphson method. A value of

x	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = 1$	$\alpha = \frac{3}{2}$
0·50	0·86600	0·96472	1·18972	1·60873
0·55	0·91039	1·01372	1·24545	1·66626
0·60	0·94990	1·05837	1·29713	1·71856
0·65	0·98488	1·09924	1·34554	1·76655
0·70	1·01565	1·13682	1·39131	1·81093
0·75	1·04145	1·17152	1·43495	1·85219
0·80	1·06546	1·20370	1·47686	1·89071
0·85	1·08480	1·23366	1·51737	1·92676
0·90	1·10045	1·26163	1·55676	1·96053
0·92	1·10564	1·27231	1·57225	1·97343
0·94	1·11018	1·28271	1·58761	1·98600
0·96	1·11402	1·29284	1·60285	1·99825
0·98	1·11704	1·30268	1·61797	2·01017
1·00	1·11887	1·31220	1·63299	2·02179
Similarity solution	1·11887	1·31220	1·63299	2·02179

TABLE 1. Values of S for mainstream $U(x) = x(1-x^2)^{-\alpha}$.

x	$\alpha = \frac{1}{5}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{1}{2}$	$\alpha = \frac{2}{3}$	$\alpha = 1$	$\alpha = \frac{3}{2}$
0·1	1·54804	1·63673	1·74356	1·84637	2·04171	2·31352
0·2	1·13831	1·25973	1·40172	1·53481	1·77977	2·10797
0·3	0·96441	1·10892	1·27378	1·42521	1·69790	2·05318
0·4	0·86497	1·02758	1·20923	1·37323	1·66330	2·02271
0·5	0·79983	0·97751	1·17219	1·34533	1·64704	2·02615
0·6	0·75366	0·94436	1·14952	1·32955	1·63910	2·02323
0·7	0·71920	0·92148	1·13529	1·32046	1·63532	2·02219
0·8	0·69247	0·90541	1·12640	1·31547	1·63365	2·02187
0·9	0·67103	0·89420	1·12120	1·31300	1·63310	2·02180
0·92	0·66721	0·89244	1·12050	1·31270	1·63306	2·02180
0·94	0·66350	0·89084	1·11992	1·31249	1·63303	2·02179
0·96	0·65987	0·88939	1·11944	1·31235	1·63301	2·02179
0·98	0·65625	0·88812	1·11910	1·31225	1·63300	2·02179
1·00	0·65228	0·88707	1·11887	1·31220	1·63299	2·02179
Similarity solution	0·65228	0·88707	1·11887	1·31220	1·63299	2·02179

TABLE 2. Values of S for mainstream $U(x) = (1-x)^{-\alpha}$.

$\Delta x = 0{\cdot}01$ was used throughout and two integrations were done in each case, the step length h in the transverse direction taking the values $h = 0{\cdot}05$ and $h = 0{\cdot}025$. Then Richardson's h^2-extrapolation formula (Smith 1965, p. 140) was used to improve the accuracy of the results. However for $\alpha = \frac{1}{5}$ the outer boundary condition had to be applied too far from the wall for these values of h and to keep computing time within bounds values of $h = 0{\cdot}1$ and $h = 0{\cdot}05$ had to be used.

From the numerical solution we can calculate the skin friction $(\partial u/\partial y)_{y=0}$, and, for comparison with the similarity solution at $x = 1$, values of

$$S = \left(\frac{2}{A_0^3}\right)^{\frac{1}{2}} (1-x)^{\frac{1}{2}(3\alpha+1)} \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

are given in tables 1 and 2 for various α for $U(x) = x(1-x^2)^{-\alpha}$ and $U(x) = (1-x)^{-\alpha}$ respectively. Here

$$A_0 = \lim_{x\to 1} (1-x)^{\alpha}\, U(x).$$

η	$x = 0{\cdot}9$	$x = 0{\cdot}99$	$x = 0{\cdot}995$	$x = 0{\cdot}9975$	$x = 1$
1	0·5175	0·4945	0·4935	0·4929	0·4922
2	0·7531	0·7161	0·7138	0·7126	0·7111
3	0·8904	0·8218	0·8181	0·8160	0·8136
4	0·9477	0·8790	0·8739	0·8701	0·8676
5	0·9773	0·9192	0·9074	0·9087	0·8990
6	0·9912	0·9370	0·9331	0·9251	0·9192
8	0·9991	0·9655	0·9588	0·9511	0·9430
10	1 0000	0·9813	0·9740	0·9664	0·9563
12		0·9903	0·9837	0·9765	0·9647
15		0·9967	0·9923	0·9863	0·9726
20		0·9996	0·9980	0·9946	0·9802
25		1·0000	0·0006	0·9981	0·9838
30			0·9999	0·9994	0·9873
35			1·0000	0·9998	0·9893
40				0·9999	0·9907
45				1·0000	0·9919
50					0·9955

TABLE 3. $\partial f/\partial\eta$ near $x = 1$ for $\alpha = \frac{1}{5}$ $(u = (1-x)^{-\alpha}\,\partial f/\partial\eta)$.

From these tables it is seen that the values of S calculated by the numerical integration of the full equations agree with those from the similarity solution (these were found by a 'shooting' technique, the equations being integrated by a Runge–Kutta method) to within the accuracy of the numerical scheme. Velocity profiles from the numerical integration were also compared with the respective similarity solution and were found to be in agreement within the expected accuracy of the two methods.

The algebraic decay for $\alpha = \frac{2}{3}$ is sufficiently rapid $[O(\eta^{-8})]$ for it to be difficult to detect the change from exponential to algebraic decay in the numerical solution for this value of α. However, for $\alpha = \frac{1}{5}$ this change is far more apparent, the decay now being $O(\eta^{-1})$, and table 3 gives the values of $\partial f/\partial\eta$ (where $u = (1-x)^{-\alpha}\,\partial f/\partial\eta$) calculated for the mainstream $U(x) = (1-x)^{-\alpha}$ by a numerical scheme which used the same method as was used for $U(x) = x(1-x^2)^{-\alpha}$, i.e. the transformation was changed at $x = \frac{1}{2}$ from Blasius variables to those appropriate to the similarity solution at $x = 1$ and a much smaller value of Δx was used near $x = 1$, so that velocity profiles could be compared at the same value of η. The values at $x = 1$ are those calculated by the numerical integration and agree with those from the similarity solution.

3. Expansion near $x = 1$

Near $x = 1$, $U(x) \sim A_0(1-x)^{-\alpha}$ and, following Goldstein (1965), we write

$$\psi = (2A_0)^{\frac{1}{2}}\xi^{\frac{1}{2}(1-\alpha)}f(\xi,\eta),$$

where $\xi = 1-x$ and $\eta = (\frac{1}{2}A_0)^{\frac{1}{2}}y/\xi^{\frac{1}{2}(1+\alpha)}$. Equations (2) and (3) become

$$\frac{\partial^3 f}{\partial\eta^3} - \frac{2}{A_0^2}U\frac{dU}{d\xi}\xi^{2\alpha+1} + (\alpha-1)f\frac{\partial^2 f}{\partial\eta^2} - 2\alpha\left(\frac{\partial f}{\partial\eta}\right)^2 = 2\xi\left(\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\,\partial\xi}\right), \tag{5}$$

with boundary conditions $f = \partial f/\partial\eta = 0$ on $\eta = 0$ and $\partial f/\partial\eta \to U(\xi)\xi^\alpha/A_0$ as $\eta \to \infty$. Taking $U(\xi) = (A_0/\xi^\alpha)(1+A_1\xi+A_2\xi^2+\ldots)$ as the expansion for $U(\xi)$, we expand $f(\xi,\eta)$ as

$$f(\xi,\eta) = f_0(\eta) + A_1\xi f_1(\eta) + \xi^2(A_2 f_2(\eta) + A_1^2 f_{21}(\eta)) + \ldots. \tag{6}$$

η	f_0'	f_1'	f_2'	f_{21}'
0	0·0000	0·0000	0·0000	0·0000
0·1	0·1873	0·2093	0·1337	0·0212
0·2	0·3456	0·3804	0·2483	0·0334
0·3	0·4773	0·5176	0·3459	0·0383
0·4	0·5854	0·6257	0·4287	0·0382
0·5	0·6732	0·7098	0·4992	0·0349
0·6	0·7437	0·7748	0·5597	0·0298
0·7	0·8000	0·8247	0·6120	0·0240
0·8	0·8446	0·8630	0·6577	0·0183
0·9	0·8798	0·8923	0·6979	0·0129
1·0	0·9073	0·9148	0·7337	0·0082
1·2	0·9455	0·9457	0·7941	0·0009
1·4	0·983	0·9645	0·8427	−0·0036
1·6	0·9818	0·9765	0·8817	−0·0062
1·8	0·9897	0·9843	0·9126	−0·0073
2·0	0·9942	0·9895	0·9366	−0·0074
2·5	0·9987	0·9963	0·9740	−0·0055
3·0	0·9997	0·9988	0·9908	−0·0031
3·5	0·9999	0·9996	0·9975	−0·0013
4·0	1·0000	1·0000	1·0000	−0·0000

TABLE 4. f_0', f_1', f_2' and f_{21}' for $\alpha = \frac{3}{2}$.

The resulting equations are (dashes denoting differentiation with respect to η)

$$f_0''' + (\alpha-1)f_0f_0'' + 2\alpha(1-f_0'^2) = 0, \tag{7}$$

$$f_n''' + (\alpha-1)f_0f_n'' + 2(2\alpha-n)(1-f_0'f_n') + (\alpha-1-2n)f_0''f_n = 0 \quad \text{for} \quad n = 1,2,\ldots \tag{8}$$

and

$$f_{21}''' + (\alpha-1)f_0f_{21}'' - 4(\alpha-1)f_0'f_{21}' + (\alpha-5)f_0''f_{21} = (3-\alpha)f_1f_1'' - 2(\alpha-1)(1-f_1'^2), \tag{9}$$

with boundary conditions $f_n(0) = f_n'(0) = 0$ $(n = 0,1,2,\ldots)$ and

$$f_n' \to 1, \quad f_{21}' \to 0 \quad \text{as} \quad \eta \to \infty. \tag{10}$$

However (10) will have to be relaxed for α in the range $0 < \alpha \leqslant 1$.

Equation (7) is the similarity solution derived by Goldstein, who showed that, for large η,

$$f_0' \sim \begin{cases} 1 + B_0 Y^{-(5\alpha-1)/(\alpha-1)} \exp\left[-\frac{1}{2}(\alpha-1)Y^2\right] & \text{for} \quad \alpha > 1, \\ 1 + B_0 Y^{-4\alpha/(1-\alpha)} & \text{for} \quad \alpha < 1, \end{cases}$$

where $Y = \eta - \delta_0$ and $\delta_0 = \lim_{\eta\to\infty} \eta - f_0(\eta)$. For $\alpha = 1$, (7) has the solution

$$f_0' = 3\tanh^2(\eta+\beta) - 2, \quad \beta = \tanh^{-1}\left(\tfrac{2}{3}\right)^{\frac{1}{2}} \tag{11}$$

(Schlichting 1960, p. 144).

3.1. *The case* $\alpha > 1$

In this case Brown & Stewartson (1965) found that the limits $\xi \to 0$ and $y \to \infty$ commute. Solutions to (8) and (9) can be found which each satisfy (10) with an exponentially small error. For $\alpha = \frac{3}{2}$ it was found that

$$f_1''(0) = 2{\cdot}2909, \quad f_2''(0) = 1{\cdot}4362 \quad \text{and} \quad f_{21}''(0) = 0{\cdot}2612,$$

and values of f_1', f_2' and f_{21}' are given in table 4.

The expansion near $\xi = 0$ is an asymptotic expansion of the type discussed by

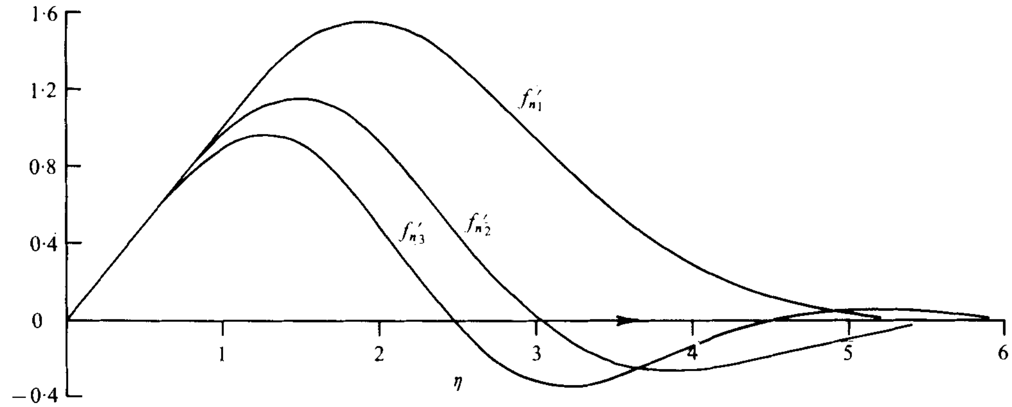

FIGURE 1. Eigensolutions f'_{n_1}, f'_{n_2} and f'_{n_3} for $\alpha = \frac{3}{2}$.

Stewartson (1957) and we therefore expect (6) to be modified by the inclusion of eigensolutions of the form $\xi^n f_n(\eta)$, for certain discrete values of n. The equation for f_n is

$$f'''_n + (\alpha-1)f_0 f''_n - 2(2\alpha - n)f'_0 f'_n + (\alpha - 1 - 2n)f''_0 f_n = 0, \tag{12}$$

with $f_n(0) = f'_n(0) = 0$ and $f'_n \to 0$ as $\eta \to \infty$. Equation (12) has a solution of the form $f_n = L_n f_a + M_n f_b$ for some constants L_n and M_n, where

$$f'_a \sim Y^{2(2\alpha-n)/(\alpha-1)}, \quad f'_b \sim Y^{-(5\alpha-1-2n)/(\alpha-1)} \exp\left[-\tfrac{1}{2}Y^2(\alpha-1)\right].$$

In general $L_n \neq 0$ and the solution is algebraic, but, for certain values of n, $L_n = 0$ and the solution will then be exponentially small. A numerical integration of (12) gives, for $\alpha = \frac{3}{2}$, the first three eigenvalues as $n_1 = 3{\cdot}5301$, $n_2 = 4{\cdot}0347$ and $n_3 = 4{\cdot}5210$ with corresponding eigensolutions f_{n_1}, f_{n_2} and f_{n_3}. Graphs of f'_{n_1}, f'_{n_2} and f'_{n_3} (normalized such that $f''_{n_i}(0) = 1$) are given in figure 1. It can be shown, using an argument similar to that described by Ackerberg (1970), that the eigenvalues are real and positive.

3.2. *The case* $0 < \alpha < 1$

Here the limits $\xi \to 0$ and $y \to \infty$ do not commute so we do not expect the solution near $\xi = 0$ to be described fully by (6). We find that this is the case, since when we attempt to solve the equations for the higher-order terms in the expansion we cannot satisfy (10) even with algebraic decay. To see this consider the behaviour of the equation $O(\xi^n)$ for large η. On neglecting smaller-order terms this is

$$f'''_n + (\alpha-1)Yf''_n - 2(2\alpha - n)f'_n = 0. \tag{13}$$

Equation (13) has a solution in terms of confluent hypergeometric functions which are $O(Y^{(5\alpha-2n-1)/(1-\alpha)}\exp[\frac{1}{2}(1-\alpha)Y^2])$ and $O(Y^{-2(2\alpha-n)/(1-\alpha)})$ respectively at infinity. The full equation for the term $O(\xi^n)$ contains a forcing term involving the previous terms in the expansion and we attempt to construct a numerical solution by first finding a particular integral ϕ_a with $\phi''_a(0) = 0$, say, and then finding a complementary function ϕ_n with $\phi''_n(0) = 1$, say. The full solution f_n will then be given by $f_n = \phi_a + \lambda\phi_n$, where the constant λ should be chosen so as to satisfy (10). However

$$f'_n \sim (K_n + \lambda L_n)\, Y^{(5\alpha-2n-1)/(1-\alpha)} \exp\left[\tfrac{1}{2}(1-\alpha)Y^2\right]$$

for some constants K_n and L_n, which means that we must take $\lambda = -K_n/L_n$, but for $n \geqslant 2\alpha$, f'_n is still algebraically large, $O(Y^{2(n-2\alpha)/(1-\alpha)})$, at infinity and the outer boundary

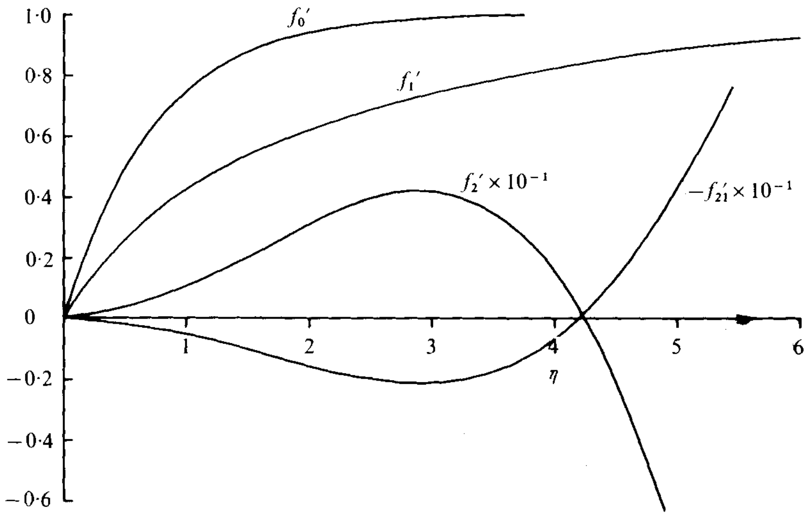

FIGURE 2. f_0', f_1', f_2' and f_{21}' for $\alpha = \frac{2}{3}$.

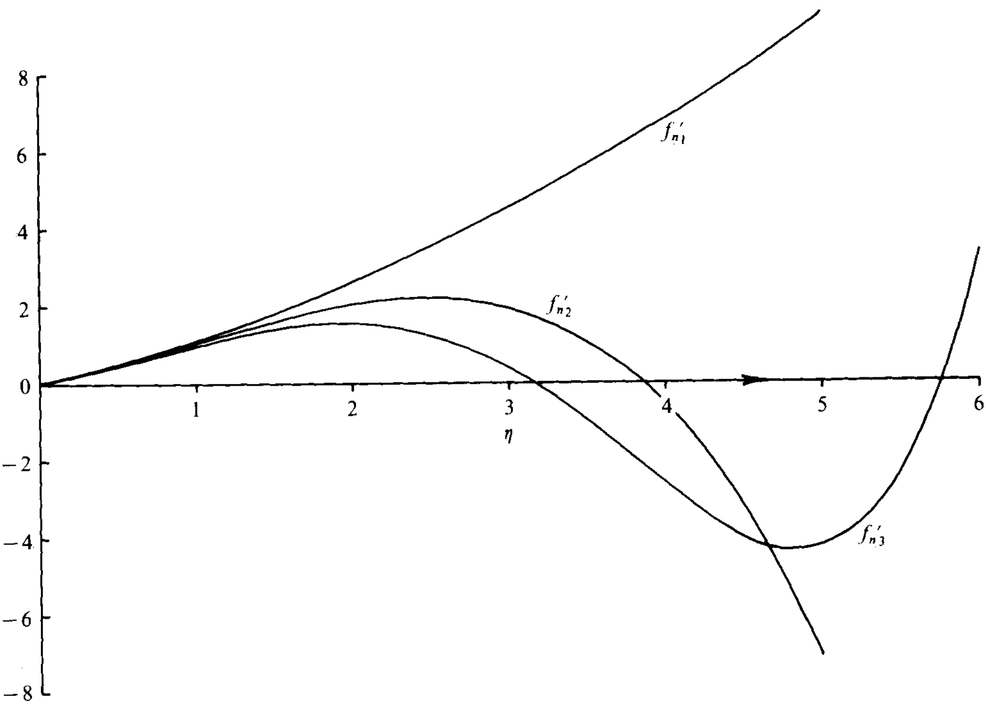

FIGURE 3. Eigensolutions f_{n_1}', f_{n_2}' and f_{n_3}' for $\alpha = \frac{2}{3}$.

condition cannot be satisfied. The breakdown of an expansion in this way when the leading term is algebraic has also been reported by Ackerberg (1970).

Equations (8) and (9) are now solved with the outer condition that the solution should not be exponentially large. For $\alpha = \frac{2}{3}$ it is found that

$$f_1''(0) = 0\cdot6956, \quad f_2''(0) = 0\cdot3693 \quad \text{and} \quad f_{21}''(0) = 0\cdot1766.$$

Graphs of f_0', f_1', f_2' and f_{21}' are given in figure 2.

Again we expect eigensolutions to appear in the expansion. These will be solutions of (12) which satisfy the relaxed outer condition. For $\alpha = \frac{2}{3}$ we find that the first three eigenvalues are $n_1 = 1\cdot5274$, $n_2 = 1\cdot8552$ and $n_3 = 2\cdot1828$. Graphs of the corresponding eigenfunctions f'_{n_1}, f'_{n_2} and f'_{n_3} (normalized such that $f''_{n_i}(0) = 1$) are given in figure 3.

As a straightforward expansion (6) cannot be constructed so as to satisfy the outer boundary condition it must be regarded as only an inner expansion. An outer expansion is then needed which approaches the mainstream with exponentially small error and which matches with the inner expansion. To do this we need to know the behaviour of the terms in (6) for large η. These asymptotic expansions contain the constant δ_0, which can be eliminated by a change in origin. As this greatly simplifies the discussion we shall assume it to have been done. We find that

$$f'_n \sim A_n + B_n h_n(\eta) + H_n(\eta), \tag{14}$$

where A_n is the coefficient of ξ^n in the expansion of $U(\xi)$, B_n is a known constant and H_n contains terms of lower order than h_n, where

$$h_n(\eta) \sim \eta^{-2(2\alpha-n)/(1-\alpha)}\left(1 - \frac{(2\alpha-n)(3\alpha-2n+1)}{(1-\alpha)^3}\frac{1}{\eta^2} + \frac{(2\alpha-n)(3\alpha-2n+1)(\alpha+1-n)(\alpha+3-2n)}{2(1-\alpha)^6}\frac{1}{\eta^4} + \dots\right) \tag{15}$$

and where, to lowest order,

$$H_0 \sim \frac{(1+3\alpha)}{2(1-5\alpha)} B_0^2 \eta^{-8\alpha/(1-\alpha)},$$

$$H_1 \sim \frac{3B_0B_1(1-2\alpha+5\alpha^2)}{(1-5\alpha)(3-5\alpha)}\eta^{2(1-4\alpha)/(1-\alpha)} - A_1B_0\frac{1+3\alpha}{1-\alpha}\eta^{-4\alpha/(1-\alpha)},$$

$$H_2 \sim \left[B_0B_2\frac{(15\alpha^2-14\alpha+15)}{5(1-\alpha)(1-5\alpha)} + \frac{B_1^2(1-3\alpha)}{2(3-5\alpha)}\right]\eta^{4(1-2\alpha)/(1-\alpha)} + A_1B_1\frac{1-3\alpha}{1-\alpha}\eta^{2(1-2\alpha)/(1-\alpha)} - A_2B_0\frac{1+3\alpha}{1-\alpha}\eta^{-4\alpha/(1-\alpha)}$$

(the asymptotic form for f'_{21} has been absorbed into that for f'_2). The eigensolutions are the same but with $A_n = 0$ and B_n indeterminate. The special cases $\alpha = \frac{1}{5}$ and $\alpha = \frac{3}{5}$ can be treated in a similar way, but now H_0, H_1 and H_2 contain terms in $\log\eta$.

The form for u at the outer edge of the inner region is then

$$u \sim A_0\xi^{-\alpha}(1 + A_1\xi + A_2\xi^2 + \dots + A_n\xi^n + \dots + \sum_n B_n h_n \xi^n + H_0 + H_1\xi + H_2\xi^2 + \dots). \tag{16}$$

The summation is over all integers and eigenvalues. The independent variable for the outer region suggested by the work of Brown & Stewartson is $\tau = (\frac{1}{2}A_0)^{\frac{1}{2}}y/\xi^{\alpha}$, and (16) suggests writing $\psi = U(\xi)y + (2A_0)^{\frac{1}{2}}\xi^{2\alpha}F(\xi,\tau)$. Equation (3) then becomes

$$\xi^{\alpha+1}U\frac{\partial^2F}{\partial\xi\,\partial\tau} - \xi^{\alpha}\left(\alpha U + \xi\frac{dU}{d\xi}\right)\left(\tau\frac{\partial^2F}{\partial\tau^2} - \frac{\partial F}{\partial\tau}\right) + \frac{A_0}{2}\xi^{1-\alpha}\frac{\partial^3F}{\partial\tau^3} - A_0\xi^2\left[2\alpha F\frac{\partial^2F}{\partial\tau^2} - \alpha\left(\frac{\partial F}{\partial\tau}\right)^2 + \xi\left(\frac{\partial F}{\partial\xi}\frac{\partial^2F}{\partial\tau^2} - \frac{\partial F}{\partial\tau}\frac{\partial^2F}{\partial\xi\,\partial\tau}\right)\right] = 0. \tag{17}$$

The outer boundary condition is that $\partial F/\partial \tau \to 0$ exponentially as $\tau \to \infty$ and the inner condition is found by writing (16) in terms of the outer variable τ. This leads to an expansion for $F(\xi, \tau)$ of the form

$$F(\xi,\tau) = F_0(\tau)+\xi^{1-\alpha}F_1(\tau)+\xi^{2(1-\alpha)}F_2(\tau)+\ldots+\xi^{2\alpha}G_1(\tau)+\xi G_2(\tau)+\ldots \qquad (18)$$

(the order of the terms in (18) depends on the particular value of α). Equation (17) then gives

$$F_1' = -\frac{1}{2(1-\alpha)}F_0''', \quad F_2' = -\frac{1}{4(1-\alpha)}F_1''', \qquad (19),(20)$$

$$G_1' = F_0F_0''-\tfrac{1}{2}F_0'^2, \quad G_2' = A_1(\tau F_0''-F_0') \qquad (21),(22)$$

(where dashes denote differentiation with respect to τ). We see that $F_0(\tau)$ cannot be determined from (17) and as

$$\sum_n B_n h_n \xi^n \sim \xi^{2\alpha}\tau^{-4\alpha/(1-\alpha)}\sum_n B_n \tau^{2n/(1-\alpha)}\left(1-\frac{(2\alpha-n)(3\alpha-2n+1)}{(1-\alpha)^3\tau^2}\xi^{1-\alpha}+\ldots\right) \qquad (23)$$

we see that each term in the inner expansion contributes to the leading term in the outer expansion. However, we do know from (23) that the behaviour of $F_0(\tau)$ for small τ is

$$F_0'(\tau) \sim \tau^{-4\alpha/(1-\alpha)}\sum_n B_n \tau^{2n/(1-\alpha)}$$

while its behaviour for large τ can be found from (1), which gives

$$F_0'(\tau) \sim \beta\frac{(1-\alpha)^{\frac{1}{2}}}{A_0^{\frac{3}{2}}}\left(\frac{2(1-\alpha)^2}{A_0}\right)^{\frac{1}{2}m}\tau^m \exp\left[-\tfrac{1}{2}(1-\alpha)\tau^2\right].$$

For $U(x) = x(1-x^2)^{-\alpha}$, $m = -3$, $A_0 = 2^{-\alpha}$ and $\beta = -3{\cdot}67$, while, for $U(x) = (1-x)^{-\alpha}$, $m = -1$, $A_0 = 1$ and $\beta = -0{\cdot}468$ (Jones & Watson 1963, p. 224). For $\alpha = \frac{2}{3}$ the numerical integration gives $B_0 = -2{\cdot}65$ and $B_1 = -0{\cdot}55$, so that

$$F_0' \sim -2{\cdot}65\tau^{-8}(1+0{\cdot}21\tau^6+\ldots)$$

for small τ.

It can also be checked that the forms for small τ of the higher-order terms F_1, F_2, G_1 and G_2 in (17) given by the matching requirement are consistent with (19), (20), (21) and (22) respectively. The special cases $\alpha = \frac{1}{5}$ and $\alpha = \frac{3}{5}$ can be treated in a similar way. Here (17) has to be modified to include terms $O(\xi^{\frac{2}{5}}\log\xi)$ and $O(\xi^{\frac{6}{5}}\log\xi)$ respectively.

3.3. *The case* $\alpha = 1$

In this case f_0 is given by (11), so though the outer boundary condition is attained with exponentially small error it is not of the same form as (1) and the limits $\xi \to 0$ and $y \to \infty$ do not commute in this case either. No trouble is encountered in solving for f_1. It is found that $f_1''(0) = 1{\cdot}4984$ and values of f_0' and f_1' are given in table 5. Again the outer boundary condition is attained with exponentially small error. When we come to consider the term $O(\xi^2)$ the expansion breaks down, again because the outer boundary condition cannot be satisfied. The equation for f_2, the term $O(\xi^2)$, is

$$f_2'''-4f_0''f_2 = 2A_1^2f_1f_1'', \qquad (24)$$

with $f_2(0) = f_2'(0) = 0$ and $f_2' \to A_2$ as $\eta \to \infty$. In general, for any numerical integration of (24),

$$f_2' \sim C_a\eta + D_a + \text{exponentially small terms},$$

η	f_0'	f_1'	ϕ_2'
0	0·0000	0·0000	0·0000
0·2	0·2872	0·2605	−0·0760
0·4	0·5016	0·4504	−0·1512
0·6	0·6561	0·5854	−0·2244
0·8	0·7649	0·6815	−0·2955
1·0	0·8403	0·7509	−0·3645
1·2	0·8920	0·8023	−0·4316
1·4	0·9272	0·8415	−0·4963
1·6	0·9510	0·8722	−0·5584
1·8	0·9671	0·8966	−0·6171
2·0	0·9779	0·9164	−0·6719
2·2	0·9852	0·9325	−0·7224
2·4	0·9900	0·9457	−0·7681
2·6	0·9933	0·9564	−0·8089
2·8	0·9955	0·9652	−0·8449
3·0	0·9970	0·9723	−0·8762
3·5	0·9989	0·9846	−0·9362
4·0	0·9996	0·9917	−0·9750
4·5	0·9999	0·9956	−0·9988
5·0	1·0000	0·9978	−1·0125
5·5	—	0·9990	−1·0196
6·0	—	0·9999	−1·0222

TABLE 5. f_0', f_1' and ϕ_2' for $\alpha = 1$.

for some constants C_a and D_a. So we can remove the term $O(\eta)$ by adding a suitably chosen complementary function to a particular integral but we then still have $f_2' \to$ constant as $\eta \to \infty$ and cannot expect this constant to be equal to A_2. Equation (24) has been solved in this way by writing $f_2 = A_1^2\phi_2$; then $\phi_2''(0) = -0{\cdot}3807$ and $\phi_2' \to -1{\cdot}0222$ so that $f_2' - A_2 \to B_2$, where $B_2 = -(A_2 + 1{\cdot}0222 A_1^2)$. Values of ϕ_2' are given in table 5.

Again (6) must be regarded as only an inner expansion and we need an outer expansion which must match with it and satisfy the outer boundary condition. To do this we find that (6) has to be modified to

$$f(\xi,\eta) = f_0(\eta) + A_1\xi f_1(\eta) + \xi^2 f_2(\eta) + \xi^2 \sum_{n=0}^{\infty} \frac{g_n(\eta)}{(-\log\xi)^{n+\frac{1}{2}}} + \dots, \tag{25}$$

where f_0, f_1 and f_2 are as before and

$$g_0''' - 4f_0'' g_0 = 0, \tag{26}$$

$$g_n''' - 4f_0'' g_n = (2n-1)(f_0'' g_{n-1} - f_0' g_{n-1}') \quad \text{for} \quad n = 1, 2, 3, \dots. \tag{27}$$

The only boundary conditions prescribed are $g_n(0) = g_n'(0) = 0$. To integrate (26) numerically we need to know $g_0''(0)$. Now for large η, $g_0' \sim C_0\eta + D_0$ for certain constants C_0 and D_0. C_0 is fixed by the matching process and this in turn determines D_0 and $g_0''(0)$. For, if we integrate (26) with $g_0''(0) = 1$, we find $g_0' \sim C_0^*\eta + D_0^*$, where $C_0^* = 2{\cdot}3263$ and $D_0^* = -2{\cdot}1810$. We then have to take $g_0''(0) = C_0/C_0^*$ and the solution of (26) is determined uniquely.

From (27) we find that, for large η,

$$g_1' \sim -\tfrac{1}{6}C_0\eta^3 - \tfrac{1}{2}D_0\eta^2 + C_1\eta + D_1, \tag{28}$$

$$g_2' \sim \tfrac{1}{40}C_0\eta^5 + \tfrac{1}{8}D_0\eta^4 - \tfrac{1}{2}C_1\eta^3 - \tfrac{3}{2}D_1\eta^2 + C_2\eta + D_2, \tag{29}$$

$$g_3' \sim -\tfrac{1}{336}C_0\eta^7 - \tfrac{1}{48}D_0\eta^6 + \tfrac{1}{8}C_1 + \tfrac{3}{8}D_1\eta^4 - \tfrac{5}{6}C_2\eta^3 - \tfrac{5}{2}D_2\eta^2 + C_3\eta + D_3 \tag{30}$$

for some constants C_1, D_1, C_2, D_2, C_3 and D_3. Now with $U(x) = A_0(1-x)^{-1}$, (1) gives

$$F(x) = -2(1-x)^2 \log(1-x)/A_0,$$

which suggests using $\zeta = (\frac{1}{2}A_0)^{\frac{1}{2}}\, y/\xi(-\log\xi)^{\frac{1}{2}}$ as the independent variable for the outer region. We thus put $\psi = U(\xi)\, y + (2A_0)^{\frac{1}{2}}\xi^2(-\log\xi)^{\frac{1}{2}}\Psi(\xi,\zeta)$, so that (3) becomes

$$\begin{aligned}&\frac{\partial^3\Psi}{\partial\zeta^3} + U\frac{\xi}{A_0}\zeta\frac{\partial^2\Psi}{\partial\zeta^2} + \xi^2(-\log\xi)\frac{2U}{A_0}\frac{\partial^2\Psi}{\partial\xi\,\partial\zeta} + \xi^2\Psi\frac{\partial^2\Psi}{\partial\zeta^2}\\ &\quad -2\xi^2(-\log\xi)\left[2\Psi\frac{\partial^2\Psi}{\partial\zeta^2} - \left(\frac{\partial\Psi}{\partial\zeta}\right)^2 + \xi\left(\frac{\partial^2\Psi}{\partial\zeta^2}\frac{\partial\Psi}{\partial\xi} - \frac{\partial^2\Psi}{\partial\zeta\,\partial\xi}\frac{\partial\Psi}{\partial\zeta}\right)\right]\\ &\quad + \frac{2}{A_0}\left(U + \xi\frac{dU}{d\xi}\right)(-\log\xi)\xi\left(\frac{\partial\Psi}{\partial\zeta} - \zeta\frac{\partial^2\Psi}{\partial\zeta^2}\right) = 0,\end{aligned} \tag{31}$$

with $\partial\Psi/\partial\zeta \to 0$ as $\zeta \to \infty$. From (28), (20) and (30) the behaviour of Ψ for small ζ is

$$\begin{aligned}\frac{\partial\Psi}{\partial\zeta} &= B_2 + C_0\left(\zeta - \frac{\zeta^3}{6} + \frac{\zeta^5}{40} - \dots\right) + \frac{D_0}{(-\log\xi)^{\frac{1}{2}}}\left(1 - \frac{\zeta^2}{2} + \frac{\zeta^4}{8} - \dots\right)\\ &\quad + \frac{C_1}{-\log\xi}\left(\zeta - \frac{\zeta^3}{2} + \frac{\zeta^5}{8} + \dots\right) + \frac{D_1}{(-\log\xi)^{\frac{3}{2}}}\left(1 - \tfrac{3}{2}\zeta^2 + \tfrac{5}{8}\zeta^4 + \dots\right) + \dots,\end{aligned} \tag{32}$$

which suggests expanding Ψ in the form

$$\Psi(\xi,\zeta) = \Psi_0(\zeta) + \sum_{n=1}^{\infty}\frac{\Psi_n(\zeta)}{(-\log\xi)^{\frac{1}{2}n}}. \tag{33}$$

The equation for Ψ_n is

$$\Psi_n''' + \zeta\Psi_n'' + n\Psi_n' = 0 \quad (n = 0, 1, 2, \dots), \tag{34}$$

with solution

$$\Psi_0' = M_0\,\mathrm{erfc}\,(\zeta/2^{\frac{1}{2}}), \quad \Psi_{n+1}' = M_n\Psi_n' \tag{35}$$

for some constants M_n. Now for small ζ,

$$\Psi_0' = M_0[1 - (2/\pi)^{\frac{1}{2}}(\zeta - \tfrac{1}{6}\zeta^3 + \dots)]$$

but from (32)

$$\Psi_0' = B_2 + C_0(\zeta - \tfrac{1}{6}\zeta^3 + \dots),$$

from which it follows that $C_0 = -B_2(2/\pi)^{\frac{1}{2}}$. Matching (35) with (32) gives $M_1 = D_0$, $M_2 = C_1$ and $M_3 = D_1$. D_0 is fixed once C_0 is, but $C_1, D_1, \dots$ are still arbitrary. For $U(x) = (1-x)^{-1}$, when $A_1 = A_2 = 0$, we have $C_0 = D_0 = 0$ but (25) will still include the terms

$$\xi^2\sum_{n=1}^{\infty}\frac{g_n(\eta)}{(-\log\xi)^{n+\frac{1}{2}}},$$

which can each be determined only to within an arbitrary multiple and which arise from the asymptotic nature of the solution (Stewartson 1957).

Finally we need to check that n takes only integer values in (25). If we include a term $\xi^2 g_m(\eta)/(-\log\xi)^m$ for any m, then $g_m''' - 4f_0'' g_m = 0$ and for large η, $g_m' \sim C_m\eta + D_m$, so that (33) would have to be modified to include the terms

$$\Psi_{m-\frac{1}{2}}(\zeta)/(-\log\xi)^{m-\frac{1}{2}} + \Psi_m(\zeta)/(-\log\xi)^m.$$

$\Psi_{m-\frac{1}{2}}$ and Ψ_m both satisfy (32) with $n = 2m-1$ and $n = 2m$, respectively, and $\Psi_{m-\frac{1}{2}} = C_m\zeta$ and $\Psi_m = D_m$ for small ζ. Equation (34) can be solved in terms of confluent hypergeometric functions (Jeffreys & Jeffreys 1962, chap. 23), from which it follows that

$$\Psi'_{m-\frac{1}{2}} = C_m\zeta\exp(-\tfrac{1}{2}\zeta^2)\,{}_1F_1(\tfrac{1}{2}(1-2m);\tfrac{3}{2};\tfrac{1}{2}\zeta^2),$$
$$\Psi'_m = D_m\exp(-\tfrac{1}{2}\zeta^2)\,{}_1F_1(\tfrac{1}{2}(1-2m);\tfrac{1}{2};\tfrac{1}{2}\zeta^2).$$

Neither of these is exponentially small for large ζ unless the series terminates. This happens when $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, which corresponds to n being an integer.

I should like to thank Dr D. B. Ingham for his helpful and constructive advice with this paper.

REFERENCES

ACKERBERG, R. C. 1965 *J. Fluid Mech.* **21**, 47.

ACKERBERG, R. C. 1970 *J. Fluid Mech.* **44**, 211.

BROWN, S. N. & STEWARTSON, K. 1965 *J. Fluid Mech.* **23**, 673.

BUCKMASTER, J. 1969 *J. Fluid Mech.* **38**, 481.

GOLDSTEIN, S. 1965 *J. Fluid Mech.* **21**, 33.

GÖRTLER, H. 1957 *J. Math. Mech.* **6**, 1.

JEFFREYS, H. & JEFFREYS, B. S. 1962 *Methods of Mathematical Physics*. Cambridge University Press.

JONES, C. W. & WATSON, E. J. 1963 *Laminar Boundary Layers* (ed. L. Rosenhead). Oxford University Press.

MERKIN, J. H. 1972 *Int. J. Heat Mass Transfer* **15**, 989.

SCHLICHTING, H. 1960 *Boundary Layer Theory*. McGraw-Hill.

SMITH, G. D. 1965 *Numerical Solution of Partial Differential Equations*. Oxford University Press.

STEWARTSON, K. 1957 *J. Math. Phys.* **36**, 173.

STEWARTSON, K. 1970 *J. Fluid Mech.* **44**, 347.